数理解析研究所講究録
第 1874 巻 2014 年 94-97

Schubert Eisenstein Series for $GL(3)$

Daniel Bump and YoungJu Choie

1 Schubert Eisenstein series

We try to explain the concept of Schubert Eisenstein series and possible arithemetic implication. This is joint work with D. Bump and a survey talk at RIMS. The original work can be found in [1]

1.1 Schubert Eisenstein series

Let us start with a general set up.

Let G be a split reductive algebraic group over a global field F. Let $\hat{T}$ be the maximal torus of the group $\hat{G}$ with opposite root data, so that $\hat{G}(\mathbb{C})$ is the connected Langlands L-group. Let $\nu\in\hat{T}(\mathbb{C})$. Then ν parametrizes a character χ_{ν} of $T(\mathbb{A})/T(F)$, where $\mathbb{A}$ is the adele ring of F. Extending χ_{ν} to the Borel subgroup $B(\mathbb{A})$, let f_{ν} be an element of the corresponding induced representation, so that

$$f_{\nu}(bg)=(\delta^{1/2}\chi_{\nu})(b)\,f(g), \qquad b\in B(\mathbb{A}). \tag{1}$$

The flag variety $X=B\backslash G$ is a projective variety. We recall its decomposition into Schubert cells. We have the Bruhat decomposition $G=\bigcup BwB$, a disjoint union over $w\in W$, and let Y_{w} be the image of BwB in X. The Schubert cell X_{w} is the Zariski closure of Y_{w}. It equals

$$\bigcup_{\substack{u\in W\\ u\leqslant w}} Y_{u},$$

where $u\leqslant w$ is the Bruhat order. Let G_{w} be the subset of G that is the union of BuB for $u\leqslant w$. It is not a subgroup. Let $X_{w}(F)$ be the set of $\gamma\in B_{F}\backslash G_{F}$ belonging to X_{w}. Thus $X_{w}(F)=B_{F}\backslash G_{w}(F)$. We may now define the *Schubert Eisenstein series*

$$E_{w}(g,f,\chi)=\sum_{\gamma\in X_{w}(F)}f(\gamma g).$$

1.2 Bott-Samelsom varieties

Let us recall the Bott-Samelson varieties and their relationship with Schubert varieties. We will denote by α_i and s_i the simple roots and corresponding simple reflections. Let $w \in W$ and let $\mathfrak{w} = (s_{i_1}, s_{i_2}, \cdots, s_{i_k})$ be a reduced decomposition of w into a product of simple reflections: $w = s_{i_1} \cdots s_{i_k}$. Let P_j be the minimal parabolic subgroup generated by B and s_j. We define a left action of B^k on $P_{i_1} \times \cdots P_{i_k}$ by

$$(b_1, \cdots, b_k) \cdot (p_{i_1}, \cdots, p_{i_k}) = (b_1 p_{i_1} b_2^{-1}, b_2 p_{i_2} b_3^{-1}, \cdots, b_k p_{i_k}). \tag{2}$$

The quotient $B^k \backslash (P_{i_1} \times \cdots \times P_{i_k})$ is the *Bott-Samelson variety* $Z_{\mathfrak{w}}$. There is a morphism $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ induced by the multiplication map that sends

$$(p_{i_1}, \cdots, p_{i_k}) \longmapsto p_{i_1} \cdots p_{i_k}.$$

This map is a surjective birational morphism.

It is known that Bott-Samelson varieties are always nonsingular, so this gives a resolution of the singularities of the Schubert variety X_w. The map $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ may not be an isomorphism. In special cases where it is an isomorphism, every element of X_w has a unique representation as a product $i_{\alpha_1}(\gamma_1) \cdots \iota_{\alpha_k}(\gamma_k)$, where if α is a root (in this case a simple root) ι_α is the Chevalley embedding of SL(2) into G corresponding to α, so the image of ι_{α_i} lies in the Levi subgroup of P_i. Beyond these special cases where $\mathrm{BS}_{\mathfrak{w}}$ is an isomorphism, in every case each element of X_w has such a factorization, and if the element is in general position, it is unique, since $\mathrm{BS}_{\mathfrak{w}}$ is birational. Let us call this a *Bott-Samelson factorization*. This means that we may write

$$E_{s_1 \cdots s_k}(g, \nu) = \sum_{\gamma_k \in B_{\mathrm{SL}_2}(F) \backslash \mathrm{SL}_2(F)} E_{s_1 \cdots s_{k-1}}(\iota_{\alpha_k}(\gamma_k) g, \nu), \tag{3}$$

building up the Schubert Eisenstein series by repeated SL_2 summations. If $\mathrm{BS}_{\mathfrak{w}}$: $Z_{\mathfrak{w}} \longrightarrow X_w$ is not an isomorphism, a modification of this method should be applicable.

1.3 GL_3 Schubert Eisenstein series, Explicit Computation

Let us be more precise : let $G = GL_3$ and let

$$\zeta^*(s) = |D_F|^{\frac{s}{2}} \prod_v \zeta_v(s), \qquad \zeta_v(s) = \begin{cases} (1 - q_v^{-s})^{-1} & \text{if } v \text{ is nonarchimedean,} \\ \Gamma_v(s) & \text{if } v \text{ is archimedean} \end{cases}$$

where we recall that D_F is the discriminant of F. For simplicity we will assume that the character χ is unramified at every place. Find $\nu_1, \nu_2 \in \mathbb{C}$ such that

$$(\delta^{1/2}\chi)\begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{pmatrix} = |y_1|^{2\nu_1+\nu_2}|y_2|^{\nu_2-\nu_1}|y_3|^{-\nu_1-2\nu_2}.$$

We will denote this character χ_{ν_1,ν_2}. Also, take $f = f^\circ$ where

$$f^\circ(g) = f^\circ_{\nu_1,\nu_2}(g) = \prod_v f^\circ_v(g_v).$$

Thus if $k \in K$

$$f^\circ_{\nu_1,\nu_2}\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} k\right) = |y_1|^{2\nu_1+\nu_2}|y_2|^{\nu_2-\nu_1}|y_3|^{-\nu_1-2\nu_2}.$$

For each $w \in W$ normalize the Schubert Eisenstein series and denote

$$E^*_w(g;\nu_1,\nu_2) = \zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)E_w(g;\nu_1,\nu_2)$$

and

$$\hat{E}^*_{s_1s_2}(\nu_1,\nu_2) = E^*_{s_1s_2}(\nu_1,\nu_2) - E^*_{s_2}(\nu_1,\nu_2) - E^*_{s_2}(\nu_2, 1-\nu_1+\nu_2)\,. \tag{4}$$

Similarly

$$\hat{E}^*_{s_2s_1}(\nu_1,\nu_2) = E^*_{s_2s_1}(\nu_1,\nu_2) - E^*_{s_1}(\nu_1,\nu_2) - E^*_{s_1}(1-\nu_1+\nu_2, \nu_1)\,. \tag{5}$$

Theorem 1 *[1] $E^*_{s_1s_2}(g;\nu_1,\nu_2)$ has meromorphic continuation to all ν_1,ν_2. It has a functional equation*

$$E^*_{s_1s_2}(g;\nu_1,\nu_2) = E^*_{s_1s_2}\left(g;\frac{2}{3}-\nu_1, \nu_1+\nu_2-\frac{1}{3}\right).$$

Moreover $\hat{E}^{**}_{s_1s_2}(g;\nu_1,\nu_2)$ is an entire function.

The similar result holds for $\hat{E}^*_{s_2s_1}(\nu_1,\nu_2)$.

1.4 Kronecker Limit Formula

Bump and Goldfeld proved the following result. If $K/\mathbb{Q}$ is a cubic field, and $\mathfrak{a}$ is an ideal class of K one may associate with $\mathfrak{a}$ a compact torus of GL_3, and if $L_{\mathfrak{a}}$ is the period of $\kappa(g)$ over this torus, then the Taylor expansion of the L-function $L(s,\mathfrak{a})$ has the form $\rho s^{-1} + L_{\mathfrak{a}} + \cdots$. Therefore if θ is a character of the ideal class group then $L(s,\theta) = \sum \theta(\mathfrak{a}) L_{\mathfrak{a}}$. The proof involves showing that the torus period of the Eisenstein series equals a Rankin-Selberg integral of a Hilbert modular Eisenstein series.

Considering Taylor expansions of E_w for various w at $\nu_1 = \nu_2 = 0$ we get

Theorem 2 *[1] We have*

$$\kappa(g) = \frac{\rho}{3}\zeta^*(2)\left[\hat{E}^{**}_{s_2 s_1}(g;0,0) + E^{**}_{s_1}(g;1,0)\right] + c_0.$$

Furthermore

$$\kappa(g) = \frac{\rho}{3}\zeta^*(2)\left[\hat{E}^{**}_{s_1 s_2}(g;1,0) + \phi_{s_2}(g)\right] + c_0'.$$

Acknowledgement I would like to thank to Prof Koji Chinen, the organizer of the RIMS conference on analytic number theory 2012 for his kind invitation. My special thanks goes to Prof Kohji Matsumoto for his kind invitation, arrangement and hospitality while I was in Nagoya and Kyoto.

This work is supported by partially supported by NRF 2012047640, NRF 2011-0008928 and NRF 2008-0061325

Address: D. Bump, Dept of Mathematics, Stanford University, CA, USA
Address: Y. Choie, Dept of Mathematics, POSTECH, Pohang, Korea

References

[1] Daniel Bump and YoungJu Choie, Schubert Eisenstein Series for $GL(3)$, Preprint (2011)